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ON HAUSDORFF DIMENSION OF SOME CANTOR ATTRACTORS

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ABSTRACT

We study what happens with the dimension of Feigenbaum-like attractors of smooth unimodal maps as the order of the critical point grows.

1. Introduction

Let f be a smooth unimodal map of an interval. We assume that f is infinitely-renormalizable with stationary combinatorics. Then f has an attractor $C(f)$ both in metric and topological senses, which is a Cantor set and which is the ω-limit set of the critical point of f. In this note we consider the following question motivated by [1], [15], and [8]: what happens with the Hausdorff dimension of $C(f)$ as the order ℓ of the critical point grows to infinity? We show that it must grow to at least $2/3$. In the orientation reversing case (which includes the classical Feigenbaum's one) we also prove that the Hausdorff dimension has a limit as ℓ tends to infinity, this limit is less than 1, and it is equal to the Hausdorff dimension of an attractor of some limit unimodal dynamics defined in [8].

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Denote by $HD(E)$ the Hausdorff dimension of a set E in $\mathbf{R}^n$.

It is well-known [9] (and follows from convergence of renormalizations) that the Hausdorff dimension $HD(C(f))$ of the attractor $C(f)$ of f depends actually only on the stationary combinatorics $\aleph$ of the map f and the criticality order ℓ of its critical point provided that ℓ is an even integer. It allows us to write $D(\aleph, \ell) = HD(C(f))$ for all smooth f with fixed $\aleph$ and ℓ.

(Note here that once the convergence of renormalizations is established for all real big enough criticalities ℓ, all results and proofs of the paper hold true for such ℓ.)

We have a priori:

$$0 < HD(\aleph, \ell) < 1. \tag{1}$$

Comment 1: (1) *If $\ell = 2$, then the upper bound in (1) can be strengthened [5]: there is a number $\sigma < 1$ such that $HD(\aleph, 2) \leq \sigma$ for all combinatorics $\aleph$.*

(2) *Feigenbaum's case $|\aleph| = 2$ with the quadratic critical point ($\ell = 2$) has been studied intensively; see [16], [17], particularly in the framework of Feigenbaum's universality [3], [4]. Numerically, $D(\aleph, 2) = 0.538\ldots$; see [6].*

(3) *Although $HD(\aleph, \ell)$ is always positive, it is not difficult to construct a sequence of stationary combinatorics $\aleph_n$ such that, for every ℓ, $HD(\aleph_n, \ell) \to 0$ as $n \to \infty$. For instance, $\aleph_n$ can be defined by the following first $n-1$ itineraries of the critical value: $n-2$ times "plus" and one time "minus". Then bounds (real or complex) imply that if $f_n(z) = z^\ell + c_n$ is infinitely-renormalizable with the stationary combinatorics $\aleph_n$, then $HD(C(f_n)) \to 0$ as $n \to \infty$.*

Note that the number $D(\aleph, 2)$ ($|\aleph| = 2$) as well as the numbers $HD(\aleph_n, \ell)$ (with fixed ℓ and big n) are less than 2/3.

Theorem 1: *For every $\aleph$,*

$$\liminf D(\aleph, \ell) > \frac{2}{3} \tag{2}$$

as ℓ tends to infinity along the even integers.

To state our result about the upper bound, we need to introduce some notions.

Non-symmetry. For a unimodal map f with a single critical point at c, denote by I_f the involution map defined in a neighborhood of c by $I_f\colon x \mapsto \hat{x}$, where $I_f(c) = c$, and otherwise $I_f(x)$ is the unique $\hat{x} \neq x$, such that $f(x) = f(\hat{x})$. If f is of the form $|E(x)|^\ell$, where $\ell > 1$ and E is a C^2-diffeomorphism, then I_f is also C^2, and $I'_f(c) = -1$. The *non-symmetry* $N(f)$ of f is said to be the number $N(f) = |I''_f(c)/2|$. It is easy to check that $N(f) = |E''(c)/E'(c)|$.

Orientation reversing combinatorics of an infinitely-renormalizable unimodal map f is such stationary combinatorics $\aleph$ that the rescaling factor of the renormalization is negative. In other words, the maps f and $f^{|\aleph|}$ have at the critical point of f different type of extrema (maximum and minimum). Examples: $|\aleph| = 2, 3$; more generally, $\aleph_n$ $(n \geq 1)$ defined in Comment 1(3).

For a combinatorial type $\aleph$ and an even integer ℓ, denote by $H_{\aleph,\ell}$ the unique universal unimodal map normalized so that $H_{\aleph,\ell}$: $[0,1] \to [0,1]$ and $H_{\aleph,\ell}(0) = 1$ (see next Section for complete definition). It is shown in [8] that the sequence $\{H_{\aleph,\ell}\}_\ell$ converges uniformly to a unimodal map $H_\aleph$: $[0,1] \to [0,1]$.

We prove in Lemma 4.3 that *if the combinatorial type $\aleph$ reverses orientation, then the sequence of non-symmetries $N(H_{\aleph,\ell})$, $\ell = 2, 4, \ldots$, is uniformly bounded.*

THEOREM 2: *For a given combinatorial type $\aleph$, assume that the sequence of non-symmetries $N(H_{\aleph,\ell})$, $\ell = 2, 4, \ldots$, is uniformly bounded. Then the Hausdorff dimension of the attractor is continuous at $\ell = \infty$: there exists*

$$\lim_{\ell\to\infty} D(\aleph, \ell) = HD(C(H_\aleph)) < 1. \tag{3}$$

Consequently, (3) holds when $\aleph$ reverses orientation.

COMMENT 2: *It is not clear if the non-symmetry $N(H_{\aleph,\ell})$ is uniformly bounded in ℓ for any type $\aleph$.*

The proof of Theorems 1 and 2 is based on recent results of [8]: see the next Section where we reduce the statements to Theorem 4.

(Note, however, that in the proof of the lower 2/3-bound we use only a part of the main result of [8], namely, the compactness (Theorem 4 in [8]).)

In turn, to prove Theorem 4 we use some results of [10], [13]; see Section 3.

From now on, we fix the type $\aleph$. Denote $p = |\aleph|$.

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2. Reduction to fixed-point maps

2.1 UNIVERSAL MAPS. For every real number $\ell > 1$, we consider a unimodal map g_ℓ: $[-1,1] \to [-1,1]$ with the critical point at 0 of order ℓ. More precisely, g_ℓ is assumed to be in the following form: $g_\ell(x) = E_\ell(|x|^\ell)$, where E_ℓ: $[0,1] \to \mathbb{R}$ is a C^2-diffeomorphism onto its image. The map $g = g_\ell$ is normalized so that

$g_\ell(0) = 1$. It is further assumed to be infinitely renormalizable with the fixed combinatorial order type $\aleph$ and to satisfy the fixed point equation:

$$\alpha g^{|\aleph|}(x) = g(\alpha x) \tag{4}$$

with $|\alpha| > 1$. By renormalization theory (see [14]), a fixed point g_ℓ for any $\ell > 1$ can be represented as $E_\ell(|x^\ell|)$ with E_ℓ a diffeomorphism in Epstein class (i.e., a diffeomorphism E of a real interval T' onto another real interval T such that the inverse map $E^{-1}: T \to T'$ extends to a univalent map $E^{-1}: (\mathbb{C} \setminus \mathbb{R}) \cup T \to (\mathbb{C} \setminus \mathbb{R}) \cup T')$.

It will be useful to deal with another unimodal map H_ℓ, which is related to g_ℓ as follows: $H_\ell(x) = |g_\ell(x^{1/\ell})|^\ell = |E_\ell(x)|^\ell$, $0 \le x \le 1$. Then H_ℓ is a unimodal map of $[0,1]$ into itself, with a strict minimum attained at some $x_\ell \in (0,1)$. It also satisfies the equation

$$\tau H^{|\aleph|}(x) = H(\tau x) \tag{5}$$

with $\tau = |\alpha|^\ell$.

We denote by $C(g_\ell)$ and $C(H_\ell)$ the attracting Cantor sets of the maps $g_\ell: [-1,1] \to [-1,1]$ and $H_\ell: [0,1] \to [0,1]$, respectively. Clearly, $HD(C(g_\ell)) = HD(C(H_\ell))$. Indeed, E conjugates $H = H_\ell$ to g restricted to $[g(1), 1]$, therefore it maps $C(H)$ to $C(g)$ and is a diffomorphism between neighborhoods of these sets.

Assume now that the order ℓ is an even integer. Then equation (4) with the normalization as above does have a unique solution, for every fixed ℓ and $\aleph$ (see [14], [11]). Consequently, $H_\ell = |g_\ell(x^{1/\ell})|^\ell$ is the unique solution of (5) with the normalization as above.

In what follows, ℓ is an even integer, and H_ℓ denotes this unique solution of (5), with its own scaling constant $\tau_\ell > 1$. (Remember that the type $\aleph$ is fixed.)

2.2 Limit dynamics. The following result is proved in [8] (even for real ℓ); see Theorems 1 and 2 and Proposition 3 there.

Theorem 3: *The sequence of maps H_ℓ converges as $\ell \to \infty$, uniformly on $[0,1]$, to a unimodal function $H = H_\infty$, which satisfies the following properties:*

1. *$\lim_{\ell\to\infty} \tau_\ell = \tau > 1$ exists, and H, τ satisfy the fixed point equation $\tau H^p(x) = H(\tau x)$ for every $0 \le x \le \tau^{-1}$. Here (as always) $p = |\aleph|$.*
2. *H has analytic continuation to the union of two topological disks U_- and U_+ and this analytic continuation will also be denoted by H.*

3. *For some $R > 1$, H restricted to either U_+ or U_- is a covering (unbranched) of the punctured disk $V := D(0,R) \setminus \{0\}$ and $\overline{U_+ \cup U_-} \subset D(0,R)$.*
4. *$U_\pm$ are both symmetric with respect to the real axis and their closures intersect exactly at x_0; $[0,x_0) \subset U_-$, $(x_0,1] \subset U_+$.*
5. *Each H_ℓ extends to a complex-analytic map defined in $U_- \cup U_+$; this sequence of analytic extensions converges to H, as $\ell_m \to \infty$, uniformly on every compact subset of $U_- \cup U_+$.*
6. *For any two open intervals I, J of the real axis, if $0 \notin J$ and $H\colon I \to J$ is one-to-one, then the branch $H^{-1}\colon J \to I$ extends to a univalent map to the slit complex plane $(\mathbf{C} \setminus \mathbf{R}) \cup J$ (this follows from the same property for H_ℓ with ℓ finite)*
7. *The mapping $G_\infty(x) := H^{p-1}(\tau^{-1}x)$ fixes x_0 and G_∞^2 has the following power series expansion at x_0:*

$$G_\infty^2(x) = x - a(x-x_0)^3 + O(|x-x_0|^4)$$

with $a > 0$.

8. *For each ℓ, the mapping $G_\ell := H_\ell^{p-1}(\tau_\ell^{-1}x)$ fixes the critical point x_ℓ of H_ℓ, $G'_\ell(x_\ell) = \pm 1/\tau_\ell^{1/\ell}$, and G_ℓ converge to G_∞ uniformly in a (complex) neighborhood of x_0.*
9. *The unimodal map $H\colon [0,1] \to [0,1]$ has a unique attractor $C(H)$, which (as for finite ℓ) is the closure of iterates of the critical point.*

2.3 The reduction. Since we know already that $HD(C(f))$ depends merely on $\aleph$ and ℓ, Theorems 1 and 2 are covered by the following statement:

Theorem 4: *The following holds.*

(a)

$$\liminf_{\ell\to\infty} HD(C(H_\ell)) \geq HD(C(H_\infty)); \tag{6}$$

(b)

$$\frac{2}{3} < HD(C(H_\infty)) < 1; \tag{7}$$

(c) *if the non-symmetries $N(H_\ell)$ are uniformly bounded as $\ell \to \infty$, then the Hausdorff dimension is continuous at infinity:*

$$\lim_{\ell\to\infty} HD(C(H_\ell)) = HD(C(H_\infty)). \tag{8}$$

The rest of the paper is devoted to the proof of this statement.

3. Background in dynamics

We prove Theorem 4 by reducing it finally to known statements about infinite conformal iterated function systems (c.i.f.s.) [10] and asymptotics near parabolic maps [13], which are given here.

3.1 C.I.F.S.. We follow [10] restricting ourself to dimension one. Let X be a closed real interval, and σ be a positive continuous function on X, which defines a new metric $d\rho = \sigma dx$ on X. Let I be a countable index set, $|I| > 1$, and let $S = \{\phi_i\colon X \to X, i \in I\}$ be a collection of injective uniform contractions w.r.t. the metric ρ: there is $\lambda < 1$, such that $\rho(\phi_i(x), \phi_i(y)) \leq \lambda\rho(x, y)$ for all i and all x, y. For every finite word $w = w_1 \cdots w_n$, denote $\phi_w = \phi_{w_1} \circ \cdots \circ \phi_{w_n}$. (Note that the metric ρ can be replaced by the Euclidean one by replacing ϕ_i by ϕ_w, where w runs over all finite words of some fixed length n s.t. $\lambda^n ||\sigma|| < 1$.) For any infinite word of symbols $w = w_1 w_2 \cdots w_j \cdots$, $w_j \in I$, denote $w|n = w_1 w_2 \cdots w_n$. The limit set L of S is $L = \bigcup_{w \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{w|n}(X)$. The system S is said to be conformal if:

(a) $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ and $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$ for all indexes $i \neq j$.

(b) There is an open set $Y \supset X$ such that all maps ϕ_i extend to $C^{1+\epsilon}$ diffeomorphisms of V into V.

(c) There is $K \geq 1$ such that $|D\phi_w(y)| \leq K|D\phi_w(x)|$ for every finite word w and all $x, y \in Y$, where $D\phi_w(x)$ means the derivative w.r.t. the metric ρ

The main object of our interest is the Hausdorff dimension of the limit set L. Note that it is the same w.r.t. the metric ρ as w.r.t. the standard Euclidean metric.

For every integer $n \geq 1$ and every $t \geq 0$ define $p_n(t) = \sum_w ||D\phi_w||^t$, where w runs over all words of length n, and $||.||$ means the sup-norm. Consequently, $P(t) = \lim_{n\to\infty} \frac{1}{n} \log p_n(t)$ is called the pressure of S at t. The parameter $\theta = \theta_S$ of the system is defined as $\inf\{t : p_1(t) < \infty\}$.

THEOREM 5: 1 (see [10], Prop. 3.3). *$P(t)$ is non-increasing on $[0, \infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$.*

2 (see [10], Thm. 3.15). *$HD(L) = \sup\{HD(L_F)\colon F \subset I$ is finite$\} = \inf\{t : P(t) \leq 0\}$; if $P(t) = 0$ then $t = HD(L)$.*

3. *If the series $p_1(\theta)$ diverges, then $P(HD(L)) = 0$ and $\theta < HD(L)$.*

(Note that clause 3 follows directly from clauses 1 and 2.)

The system with $P(t) = 0$ is called **regular**. The system is regular if and only if there is a t-**conformal measure**, i.e., a probability measure m such that $m(L) = 1$ and, for every Borel set $A \subset X$ and every $i \in I$, $m(\phi_i(A)) =$

$\int_A |D\phi_i|^t dm$ and $m(\phi_i(X) \cap \phi_j(X)) = 0$ for all $i \neq j$ from I.

3.2 Dominant convergence and forward Poincaré series. Here we follow [13], adapting the statements slightly for our applications.

Let $f_n: U \to \mathbf{C}$ be a sequence of holomorphic maps which converges uniformly in a topological disk U of the plane to a holomorphic map $f: U \to \mathbf{C}$. Assume that $c_n \to c \in U$, and the following expansions hold:

$$f_n(z) = c_n + \lambda_n(z - c_n) + b_n(z - c_n)^2 - a_n(z - c_n)^3 + \cdots,$$

where $0 < \lambda_n < 1$, $b_n, a_n \in \mathbf{R}$, and

$$f(z) = z - a(z - c)^3 + \cdots,$$

where $a > 0$, i.e., f is parabolic with two ("real") attracting petals at c. (In particular, $b_n \to 0$ and $a_n \to a$.) Then f_n is said to converge to f **dominantly**, if there is $M > 0$ such that $|b_n| \leq M|\lambda_n - 1|$ for all n.

For every $g = f_n$ and $t > 0$ define the (forward) Poincaré series

$$P_t(g, x) = \sum_{i \geq 0} |(g^i)'(x)|^t$$

and, for any open set $V \subset U$, define

$$P_t(g, V, x) = \sum_{g^i(x) \in V} |(g^i)'(x)|^t.$$

We say the Poincaré series for (f_n, t_n) converge uniformly if, for any compact set K ($c \notin K$) in an attracting petal of f, and any $\epsilon > 0$, there exists a neighborhood V of c such that $P_{t_n}(f_n, V, x) < \epsilon$ for all n large enough and all $x \in K$. We will need

Theorem 6: *Let f_n, f be as above, and $t_n \to t > 2/3$. If $f_n \to f$ dominantly, then the Poincaré series for (f_n, t_n) converge uniformly.*

This is a particular case of Theorem 10.2 proven in [13]. For completeness, we give a short proof of Theorem 6; see Appendix.

4. Proof of Theorem 4

4.1 Presentation system for the Cantor attractor. We repeat (with modifications) a construction from [8] (cf. [7], [2]), which is crucial for our proof. Let H be either one of H_ℓ or the limit map H_∞. Consequently, let G be either the corresponding G_ℓ or G_∞. We construct the presentation system for the attractor $C(H)$, which is an infinite iterated function system Π on an interval I so that $C(H) \cap I$ is (up to a countable set) the limit set of Π. Moreover, this picture converges, as $\ell \to \infty$, to the corresponding picture of the limit map.

Denote $c_j = H^{j-1}(0)$, $j \geq 0$, the j-iterate of the critical point c_0 of H (i.e., $c_0 = x_\ell$ for $H = H_\ell$ and $c_0 = x_0$ for $H = H_\infty$). Let $I = [c_p, c_{2p}]$. Then we define a sequence of maps $\psi_{k,m}: I \to I$, $k = 1, 2, \ldots$, $m = 1, 2, \ldots, p-1$, as follows. Let $H^{-(p-m)}: [c_p, c_{2p}] \to [c_m, c_{p+m}]$ denote the corresponding one-to-one branch of $H^{-(p-m)}$. Then set

$$\psi_{k,m} = G^k \circ H^{-(p-m)}. \tag{9}$$

LEMMA 4.1: (a)

$$I_{k,m} := \psi_{k,m}(I) = [c_{p^k m}, c_{p^k (p+m)}] \subset I.$$

The intervals $I_{k,m}$ are pairwise disjoint.

(b) *Let L be the limit set of the system $\{\psi_{k,m}\}$ (in other words, L is the set of non-escaping points of the inverse maps $\psi_{k,m}^{-1}: I_{k,m} \to I$). Then the closure $\overline{L} = L \cup P$, where P is a subset of pre-images of the critical point c_0, and*

$$\overline{L} = C(H) \cap I.$$

Proof: From the functional equation for H, $G(c_j) = c_{pj}$, $j \in \mathbf{Z}$, where c_j, for $j < 0$, is an H^j-preimage of c_0. The rest follows. ∎

Denote by $\Pi_\ell = (\psi_{k,m}^{(\ell)})_{k,m}$, resp. $\Pi_\infty = (\psi_{k,m}^{(\infty)})_{k,m}$, the presentation system of H_ℓ, resp. H_∞.

The notation $B(E)$ stands for the round disk which is based on an interval $E \subset \mathbf{R}$ as a diameter.

LEMMA 4.2: *Let $\Pi = \{\psi_{k,m}: I \to I_{k,m}\}_{k,m}$ be either Π_ℓ or Π_∞.*

(1) *There exists a fixed open interval J, which contains I for all ℓ large enough (including $\ell = \infty$), such that each $\psi_{k,m}$ extends to a univalent map $\psi_{k,m}: B(J) \to B(J_{k,m})$, where $J_{k,m} = \psi_{k,m}(J)$ are pairwise disjoint intervals properly contained in J.*

Therefore, there is $\lambda < 1$ (dependent only on the type $\aleph$) such that $||D\psi_{k,m}||_\rho < \lambda$, for all k, m, and $\ell \leq \infty$ large enough, where $||D\psi_{k,m}||_\rho$ denotes the supremum on the interval I of the derivative of $\psi_{k,m}$ in the hyperbolic metric ρ of $B(J)$.

(2) Π *(with the metric ρ restricted to the closed subinterval I of J) is an infinite conformal iterated function system, such that:*

(a) $\theta_{\Pi_\ell} = 0$ *for* $\ell < \infty$;
(b) $\theta_{\Pi_\infty} = 2/3$, $P(\theta_{\Pi_\infty}) = \infty$;
(c) Π_ℓ, $\ell \leq \infty$, *is regular.*

Proof: (1) follows from Theorem 3 (7), and from another representation of the maps of the system: $\psi_{k,m} = H^{-1} \circ \tau^{-k} \circ H^{-(p-m-1)}$ which is a consequence of the equation $H \circ G = \tau^{-1} \circ H$. (2a) is immediate because c_0 is the attracting fixed point of G for finite ℓ.

(2b)–(2c). Since $G = G_\infty$ has a neutral fixed point with two attracting petals, and $\psi'_{k,m}(x) = (G^k)'(H^{-(p-m)}(x))(H^{-(p-m)})'(x)$, we obtain the following asymptotics, as $k \to \infty$, for the presentation system: $|\psi'_{k,m}(x)|/k^{-3/2} \to a_m(x)$ where, for fixed $m = 1, \ldots, p-1$, the function $a_m(x)$ is continuous and positive on I. It follows from here that the critical exponent θ of the system is $\theta = 2/3$. Thus, $p_1(\theta) = \infty$ for all $\ell \leq \infty$. Hence, by Theorem 5, the system $\{\psi_{k,m}\}$ is regular. ∎

4.2 Hausdorff dimension for the limit map. As a corollary, we obtain Theorem 4, (a)–(b):

Corollary 4.1: (1) $2/3 < HD(C(H_\infty)) < 1$,
(2) $\liminf_{\ell\to\infty} HD(C(H_\ell)) \geq HD(C(H_\infty)) > \frac{2}{3}$.

Proof: Denote $H = H_\infty$. Since H is regular and $P(2/3) = \infty$, then $HD(C(H)) > 2/3$. On the other hand, the Lebesgue measure of $I \setminus \bigcup_{k,m} I_{k,m}$ is positive. Therefore ([10], Theorem 4.5), $HD(C(H)) = HD(C(H) \cap I) = HD(L) < 1$.

(2) follows from Theorem 5: for every $\delta > 0$, there is a finite subsystem F_∞ of Π_∞ with the Hausdorff dimension of its limit set at least $HD(C(H_\infty)) - \delta$. Since the corresponding finite subsystem F_ℓ converges to F_∞ as $\ell \to \infty$, then the Hausdorff dimension of the limit set of F_ℓ is at least $HD(C(H_\infty)) - 2\delta$, for all ℓ large enough. The result follows. ∎

4.3 Non-symmetry and dominant convergence. It remains to prove Theorem 4 (c).

Denote $\epsilon = 1$ or 2 depending on whether $G'_\infty(x_0) = 1$ or -1.

Lemma 4.3: 1. *The sequence* G^ϵ_ℓ *converges to* G^ϵ_∞ *dominantly if and only if the sequence of non-symmetries* $N(H_\ell)$ *is bounded.*

2. *If the combinatorics reverses orientation, then* G^2_ℓ *converges dominantly to* G^2_∞, *and the non-symmetries* $N(H_\ell)$ *are uniformly bounded.*

Proof: Let $H = H_\ell$ and $G = G_\ell$, $\tau = \tau_\ell$, and $I = I_H$. We have: $H(G(I(x))) = \tau^{-1}H(I(x)) = \tau^{-1}H(x) = H(G(x))$, i.e., $I \circ G = G \circ I$. The latter equation gives us: $|(G^\epsilon)''(x_\ell)| = N(H)\lambda(1-\lambda)$, where $\lambda = \lambda_\ell = (G^\epsilon)'(x_\ell) \in (0,1)$. This implies 1.

To prove 2, notice that the combinatorics reverses orientation if and only if $G'_\infty(x_0) = -1$. Then we get the dominant convergence, because $|(G^2)''(x_\ell)| = |G''(x_\ell)||\lambda|(1-|\lambda|)$ and $G''(x_\ell) = G''_\ell(x_\ell)$ converges to the number $G''_\infty(x_0)$, as $\ell \to \infty$. (One can also refer formally to [13], Proposition 7.3.) ∎

4.4 CONFORMAL MEASURES OF THE PRESENTATION SYSTEMS. Remember that $\Pi_\ell = (\psi^{(\ell)}_{k,m}\colon I^\ell \to I^\ell_{k,m})_{k,m}$, resp. $\Pi_\infty = (\psi^{(\infty)}_{k,m}\colon I^\infty \to I^\infty_{k,m})_{k,m}$, the presentation system of H_ℓ, resp. H_∞. We know that Π_ℓ, Π_∞ are regular. Denote by μ_ℓ, resp. μ_∞, the unique probability h_ℓ-conformal, resp. h_∞-conformal, measure of Π_ℓ, resp. Π_∞, where $h_\ell = HD(C(H_\ell) \cap I^\ell) = HD(C(H_\ell))$, $h_\infty = HD(C(H_\infty) \cap I^\infty) = HD(C(H_\infty))$. (Notice that the measures have nothing to do with conformal measures of H_ℓ, H_∞, because the dynamics are completely different.) Since any regular system has a unique conformal measure, to prove that $h_\ell \to h_\infty$ it is enough to prove that a weak limit ν of a subsequence of μ_ℓ is a conformal measure of Π_∞. For this to be true, it is enough to check that the support of ν is contained in the limit set L_∞ of Π_∞. Note that by Lemma 4.1(b), the set $\overline{L}_\infty \setminus L_\infty$ is countable. Therefore, it is enough to prove that ν has no atoms. Thus Theorem 4(c) follows from

LEMMA 4.4: *If the non-symmetries $N(H_\ell)$ are uniformly bounded, then the measure ν has no atoms.*

Proof: Let the point $a \in \operatorname{supp}(\nu) = \overline{L}_\infty$, where L_∞ is the limit set of Π_∞, be an atom of ν. Then there is $\sigma > 0$ such that for all $r > 0$ small enough, $\mu_\ell(B(a,r)) > \sigma$ along a subsequence of ℓ's. Since $\psi_{k,m}$ are uniform contractions and the measures are probabilities, one sees that $a \in \overline{L}_\infty \setminus L_\infty$, i.e., after all, one can assume that $a = x_0$. Now $\mu_\ell(B(x_0,r)) \le \sum_{I^\ell_{k,m} \cap B(x_0,r) \neq \emptyset} \int_{I^\ell} |D\psi^{(\ell)}_{k,m}|^{h_\ell} d\mu_\ell \le C \sum |(G^k_\ell)'(y_{\ell,m})|^{h_\ell}$, for some fixed $C > 0$, some points $y_{\ell,m}$ from a fixed compact set K, $x_0 \notin K$ (if ℓ is big enough), and the latter sum runs over such k that $G^k_\ell(y_{\ell,m}) \in B(x_0, r')$, where $r' \to 0$ as $r \to 0$. Then a contradiction follows directly from Lemma 4.3 and Theorem 6 (note that $t > 2/3$ by Corollary 4.1(2)). ∎

5. Appendix: Proof of Theorem 6

1. If $h_n \to h$ is a sequence of injective holomorphic maps in a fixed neighborhood of c, which converges to an injective h uniformly, then the Poincaré series for (f_n, t_n) converge uniformly iff the Poincaré series for $(h_n \circ f_n \circ h_n^{-1}, t_n)$ converge uniformly. In particular, one can assume that $c_n = c = 0$.

2 (see Theorem 7.2 of [13]). Let $h_n(z) = z - B_n z^2$, where

$$B_n = b_n/(\lambda_n(\lambda_n - 1)).$$

Since $|b_n| \le M|\lambda_n - 1|$ for all n, there is a subsequence of h_n as in Step **1**. On the other hand, $h_n \circ f_n \circ h_n^{-1}(z) = \lambda_n z + O(z^3)$. It means one can assume that $f_n(z) = \lambda_n - a_n z^3 + \cdots$ where $a_n \to a > 0$, $0 < \lambda_n < 1$ and $\lambda_n \to 1$.

3. For f_n, make a change $z = \hat{h}_n(w) = d_n w^{-1/2}$, where

$$w \in F = \{w : \mathrm{Re}(w) > R_0\} \quad \text{and} \quad d_n = (\lambda_n^3/(2a_n))^{1/2}.$$

For $g_n = \hat{h}_n^{-1} \circ f_n \circ \hat{h}_n$, we have $g_n(w) = \sigma_n w + 1 + \alpha_n(w)$, where $\sigma_n = \lambda_n^{-2} > 1$ and $\sigma_n \to 1$, α_n converge uniformly in F to the corresponding α for $g = \hat{h}^{-1} \circ f \circ \hat{h}$, $\hat{h} = \lim \hat{h}_n$, and $\alpha_n(w) = O(|w|^{-1/2})$, $\alpha(w) = O(|w|^{-1/2})$.

To deal with g_n^i, we prove the following simple Claim. This is weaker than Theorems 8.1–8.3 of [13], but still enough for our needs.

CLAIM 1: *For every $\delta > 0$ there is $R_\delta > R_0$ and, for every n, there is a $1 + \delta$-quasiconformal map ϕ_n of the plane that fixes $0, 1$, and ∞ such that $\phi_n^{-1} \circ g_n \circ \phi_n = T_n$, where $T_n(w) = \sigma_n w + 1$, for $\mathrm{Re}(w) > R_\delta$. Passing to a subsequence, one can assume that $\phi_n \to \phi$, so that $\phi^{-1} \circ g \circ \phi = T$, $T(w) = w + 1$.*

Proof: Fix $\delta > 0$. Denote $\Pi(R_1, R_2) = \{w : R_1 < \mathrm{Re}(w) < R_2\}$. Then $|\alpha_n(w)|$ and $|\alpha_n'(w)| \le \sup\{|\alpha_n(t)| : |t - w| < 1\}$ are uniformly arbitrarily small as $w \in L := \{\Re(w) = R_\delta\}$ and $R_\delta \to \infty$. Therefore, all $\sigma_n w$ can be joined to $z(w) := \sigma_n w + 1 + \alpha_n(w)$ by disjoint intervals $I(w)$ in the strip between $\sigma_n L$ and $z(L)$. The mapping ϕ_n, which is affine on each interval $[\sigma_n w, \sigma_n w + 1]$ onto $I(w)$ together with the identity on $\Pi(R_\delta, \sigma_n R_\delta)$, is $1 + \delta$ quasi-conformal on $\Pi(R_\delta, \sigma_n R_\delta + 1)$. Then we extend ϕ_n to $\mathrm{Re}(w) > \sigma_n R_\delta + 1$ by the (conformal) dynamics of g_n, T_n, and define it identity on the rest of the plane.

CLAIM 2: *For every real $p > 1$, there is M such that $|(T_n^i)'(w)|/|T_n^i(w)|^p \le M i^{-p}$ for all i, n, and all $w > 1$.*

Indeed, denote $C(i, n) = \sigma_n^i$. Consider any subsequence (i_j, n_j), $j \to \infty$. If $C(i, n)$ is bounded from above along this subsequence, then applying as in [12], Section 6, the inequality between arithmetic and geometric means, we can write $T_n^i(w) = \sigma_n^i w + (1+\sigma_n+\cdots+\sigma_n^{i-1}) \geq (i+1)w^{1/(i+1)}\sigma_n^{i/2} \geq C(i,n)^{1/2}i$, so that $|(T_n^i)'(w)|/|T_n^i(w)|^p \leq C(i,n)^{1-p/2}i^{-p} = O(i^{-p})$ along the subsequence. If now $C(i,n) \to \infty$ along (i_j, n_j) (and $\sigma_{n_j} \to 1$), then

$$\frac{|(T_n^i)'(w)|}{|T_n^i(w)|^p} = \frac{|\sigma_n^i|}{|\sigma_n^i w + (\sigma_n^i - 1)/(\sigma_n - 1)|^p}$$
$$\sim C(i,n)|\sigma_n - 1|^p/C(i,n)^p \sim (\log C(i,n))^p/C(i,n)^{p-1}i^{-p} = o(i^{-p}).$$

4. From Steps 1–2, Claim 1, and the Koebe distortion theorem, it follows that it is enough to prove the theorem assuming that the compact K is a point x, which, moreover, lies on an attracting direction of f, and small neighborhood V can be replaced by big indexes. We have: $|(f_n^i)'(x)| = K|(g_n^i)'(w)|/|g_n^i(w)|^{3/2}$, where $K > 0$ and $w > R$ depend only on $x > 0$. Thus we need to show that, if $t_n \to t > 2/3$, for a given $w > 0$ close enough to $+\infty$, for any $\epsilon > 0$ there exists an index i_0 such that $S(g_n, i_0, t_n) := \sum_{i \geq i_0} |(g_n^i)'(w)/g_n^i(w)^{3/2}|^{t_n} < \epsilon$ for all n large enough. Claim 2 (with $p = 3/2$) implies immediately that this is true for $g_n = T_n$.

To handle $S(g_n, i_0, t_n)$ in general, we compare it with $S(T_n, i_0, t_n)$ and proceed similar to [13], Section 10. Due to the Koebe distortion theorem, one can replace the derivative by the ratio of diameters. By Claim 1, the change of the diameters when passing from g_n to T_n is Hölder with the exponent arbitrarily close to 1. Then we apply Claim 2 with p arbitrarily close to 3/2.

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